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The Distribution and Average Order of the Coefficients of Dedekind ζ Functions

JAMES L. HAFNER*

Department of Mathematics, Institute for Advanced Study, Princeton, New Jersey 08540
and Department of Mathematics, California Institute of Technology, Pasadena, California 91125

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Let $K/\mathbb{Q}$ be an algebraic number field and $\zeta_K(s)$ be the associated Dedekind ζ function. A quantitative estimate is proved which shows that the average order of the coefficients of $\zeta_K^m(s)$ (for $m \in \mathbb{Z}^+$) arises from infrequent occurrences of very large values of these coefficients. This leads to new Ω-estimates for the associated error terms, improving results of Szegö and Walfisz.

Let $K/\mathbb{Q}$ be any algebraic number field of degree $N = r_1 + 2r_2 > 1$ where r_1 is the number of real conjugates and $2r_2$ is the number of imaginary conjugates of K. Let $\zeta_K(s) = \sum_{n=1}^{\infty} a(n)\, n^{-s}$ be the Dedekind ζ function associated with K. For a positive integer m, write

$$\zeta_k^m(s) = \sum_{n=1}^{\infty} d_m(n, K)\, n^{-s}.$$

Note that $d_m(n, K) \geqslant 0$ and $d_1(n, K) = a(n)$ for all n. It is known (see, for example, Chandrasekharan and Narasimhan [2]) that for $\rho \geqslant 0$,

$$\sum_{n \leqslant x} d_m(n, K)(x-n)^{\rho} = x^{1+\rho} P_{m-1}^{\rho}(\log x) + \Delta_{m,\rho}(x) \tag{1}$$

where $\Delta_{m,\rho}(x) = O(x^{1+\rho-1/(mN)})$ and P_{m-1}^{ρ} is a polynomial of degree $m-1$ with coefficients depending on m, K and ρ. In this paper, we prove the following two theorems on the distribution of $d_m(n, K)$ and on the order of $\Delta_{m,\rho}(x)$.

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THEOREM 1. *There exists a positive number A depending only on K such that if $A(m) = m \log m + mA$, $B = B(x) > 0$ is any function of x and $L_m(x, B) = A(m) \log\log x - B\sqrt{\log\log x}$, then*

$$\sum_{\substack{n \leqslant x \\ d_m(n,K) \leqslant \exp L_m(x,B)}} d_m(n, K) \leqslant \frac{c_0}{B^2} x(\log x)^{m-1} \tag{2}$$

for some constant c_0 independent of x and B and

$$\sum_{\substack{n \leqslant x \\ d_m(n,K) \geqslant \exp L_m(x,B)}} 1 \ll x(\log x)^{m-1-A(m)} \exp(B\sqrt{\log\log x}). \tag{3}$$

The number A is given explicitly in terms of the Dirichlet density of certain classes of primes. See Eq. (11). In particular if $K/\mathbb{Q}$ is Galois then $A = \log N$.

Theorem 1 shows that the main contribution of the summatory function in (1) (with $\rho = 0$) arises from the very large values of $d_m(n, K)$, those values much larger than their average order $(\log n)^{m-1}$. From (3) it is evident that there are relatively few of these values.

THEOREM 2. *For $\rho \geqslant 0$, put $\theta = (mN-1)(2\rho+1)/(2mN)$, $\kappa = (mN-1-2\rho)/(2mN)$ and $\tau = \kappa(A(m)-m+1)+m-1$. If $0 \leqslant \rho < (mN-1)/2$, then there exists a positive number B depending on m, K and ρ such that*

$$\Delta_{m,\rho}(x) = \Omega^*\{x^\theta(\log x)^\kappa(\log\log x)^\tau \exp(-B\sqrt{\log\log\log x})\}$$

where

$$\begin{aligned}
\Omega^* &= \Omega_+ &&\text{if } mr_2 \equiv 0 \pmod 4 \text{ and } \rho \geqslant (mN-3)/2\\
&= \Omega_+ &&\text{if } mr_2 \equiv 3 \pmod 4 \text{ and } \rho \geqslant (mN-2)/2\\
&= \Omega_- &&\text{if } mr_2 \equiv 2 \pmod 4 \text{ and } \rho \geqslant (mN-3)/2\\
&= \Omega_- &&\text{if } mr_2 \equiv 1 \pmod 4 \text{ and } \rho \geqslant (mN-2)/2\\
&= \Omega_\pm &&\text{otherwise.}
\end{aligned}$$

The results in Theorem 2 are new. In [7], Szegö and Walfisz obtained an Ω-result for $\Delta_{m,0}(x)$ but with $\tau = m-1$ and $B = 0$. In [3], the author generalized their result to $0 \leqslant \rho < (mN-1)/2$ with the same τ and B. Later in [5], the author further improved the result by putting $\tau = (m\log m - m + 1) + m - 1$ and some $B > 0$. Theorem 2 then increases the value of τ by the addition of the term $\kappa mA > 0$. This theorem appears to be the best that can be obtained by present methods.

Furthermore, the above theorems are valid if $K=\mathbb{Q}$, the rationals ($N=1$) but with $A=0$. This case was already reported by the author in [3, 5].

The basic outline of the proofs is as follows. We first specify and describe the classes of primes which determine the constant A. We define an arithmetical function $W_m(n)$ which counts the number of distinct prime factors of n, each prime being weighted according to the class in which it lies. We then prove a lemma on the joint distribution of $W_m(n)$ and $d_m(n, K)$. Theorem 1 then follows easily from the lemma. Theorem 2 is a direct consequence of Theorem 1 and a general Ω-theorem of the author. A general description of this theorem is given in the sequel.

We shall assume that K and m are fixed. Thus all constants and O-estimates may depend on K and m in some unspecified way. We shall also write $d_m(n)$ for $d_m(n, K)$ and hope that no confusion arises between our $d_m(n)$ and the usual m-fold divisor function on the ordinary integers.

1. The Classes of Primes

For $\nu=0, 1, 2,..., N$, let P_ν be the set of rational primes which are unramified in K and which are divisible by exactly ν primes of K of residual degree equal to 1. Let P^* be the ramified primes. Note that P^* has only finitely many elements and P_{N-1} is empty. Also if $K/\mathbb{Q}$ is Galois then P_ν is empty if $\nu=1, 2,..., N-1$. The P_ν are disjoint and together with P^* exhaust all the rational primes. Furthermore, if p is in P_ν then $\nu=a(p)$, the number of ideals of K with norm p and

$$\nu m = d_m(p). \tag{4}$$

We can describe the sets P_ν in another way. Let L be the Galois closure of K, $G=\mathrm{Gal}(L/\mathbb{Q})$ and $H=\mathrm{Gal}(L/K)$. For any rational prime p, let $C(p)=[(L/\mathbb{Q})/p]$ be the conjugacy class of Frobenius automorphisms $(\mathfrak{p}, L/\mathbb{Q})$ for primes $\mathfrak{p}$ of L above p. Then

$$P_\nu=\{p\colon \text{for each } \tau\in C(p),\ |\{\sigma\in G\colon \tau\in\sigma H\sigma^{-1}\}|=\nu\cdot|H|\}.$$

By the prime ideal theorem of Artin [1, Satz 4], if P_ν is not empty then there exists a positive constant δ_ν such that

$$\sum_{\substack{p\leqslant x\\ p\in P_\nu}} 1=\frac{\delta_\nu x}{\log x}+O\left(\frac{x}{\log^2 x}\right). \tag{5}$$

The constants δ_ν are the Dirichlet density of the sets P_ν and satisfy

$$\sum_{\nu=1}^{N} \nu\cdot\delta_\nu=1 \tag{6}$$

and

$$\delta_v = |G|^{-1} \cdot |\{\tau \in G: |\{\sigma \in G: \tau \in \sigma H \sigma^{-1}\}| = |H| \cdot v\}|. \tag{7}$$

As a simple consequence of (5) we have

$$\sum_{\substack{p \leqslant x \\ p \in P_v}} \frac{1}{p} = \delta_v \log\log x + O(1). \tag{8}$$

We remark that the estimate in (5) is weaker than Artin's theorem but (5) suffices to yield (8) which is all we need here.

2. The Lemma

For each $v = 1, 2, \ldots, N$ and n a positive integer, let $\omega_v(n)$ be the number of distinct primes in P_v which divide n. Put

$$W_m(n) = \sum_{v=1}^{N} \omega_v(n) \log vm. \tag{9}$$

This is the weighted prime counting function mentioned in the introduction. For all n, if $d_m(n) > 0$ then

$$d_m(n) \geqslant \exp W_m(n). \tag{10}$$

This follows from the fact that $d_m(n)$ is multiplicative and

$$d_m(p^\alpha) \geqslant (-1)^\alpha \binom{-vm}{\alpha} \geqslant vm = \exp W_m(p^\alpha) \quad \text{for } \alpha \geqslant 1 \text{ and } p \text{ in } P_v.$$

Lemma. *Put*

$$A = \sum_{v=1}^{N} \delta_v v \log v \tag{11}$$

and $A(m) = m \log m + mA$. Then

$$\sum_{n \leqslant x} d_m(n)\{W_m(n) - A(m)\log\log x\}^2 = O(x(\log x)^{m-1} \log\log x).$$

Proof. First, we have from (1), with $\rho = 0$,

$$\sum_{n \leqslant x} d_m(n) = c_1 x(\log x)^{m-1} + O(x(\log x)^{m-2}) \tag{12}$$

where c_1 is the leading coefficient of $P^0_{m-1}(x)$.

Second, by (4), (8) and (12), for each $\nu = 1, 2, \ldots, N$,

$$\begin{aligned}
&\sum_{n \leqslant x} d_m(n)\, \omega_\nu(n) \\
&\quad = \sum_{\substack{p \leqslant x \\ p \in P_\nu}} \sum_{\substack{n \leqslant x \\ p \mid n}} d_m(n) \\
&\quad = \nu m \sum_{\substack{p \leqslant x \\ p \in P_\nu}} \sum_{n \leqslant x/p} d_m(n) + O\Bigg(\sum_{\substack{p \leqslant x \\ p \in P_\nu}} \sum_{\alpha \geqslant 2} d_m(p^\alpha) \sum_{n \leqslant x/p^\alpha} d_m(n) \Bigg) \\
&\quad = \nu m c_1 \sum_{\substack{p \leqslant x \\ p \in P_\nu}} \frac{x}{p} (\log x/p)^{m-1} + O\Bigg(x(\log x)^{m-1} \sum_{\substack{p \leqslant x \\ p \in P_\nu}} \sum_{\alpha \geqslant 2} \frac{d_m(p^\alpha)}{p^\alpha} \Bigg) \\
&\quad = \delta_\nu \nu m c_1 x (\log x)^{m-1} \log\log x + O(x(\log x)^{m-1}).
\end{aligned}$$

Thus, using (6), (9) and (11),

$$\sum_{n \leqslant x} d_m(n)\, W_m(n) = c_1 A(m)\, x(\log x)^{m-1} \log\log x + O(x(\log x)^{m-1}). \tag{13}$$

Similarly, we show that

$$\begin{aligned}
\sum_{n \leqslant x} d_m(n)\, W_m^2(n) = A^2(m)\, c_1 x(\log x)^{m-1} (\log\log x)^2 \\
+ O(x \log x)^{m-1} \log\log x).
\end{aligned} \tag{14}$$

This requires only

$$\sum_{\substack{pq \leqslant x \\ p \in P_\nu, q \in P_\mu}} \frac{1}{pq} = \delta_\nu\, \delta_\mu (\log\log x)^2 + O(\log\log x)$$

which can be proved directly from (8) and the fact that

$$\sum_{\substack{p \leqslant \sqrt{x} \\ p \in P_\nu}} \frac{1}{p} \sum_{\substack{q \leqslant \sqrt{x} \\ q \in P_\mu}} \frac{1}{q} \leqslant \sum_{\substack{pq \leqslant x \\ p \in P_\nu, q \in P_\mu}} \frac{1}{pq} \leqslant \sum_{\substack{p \leqslant x \\ p \in P_\nu}} \frac{1}{p} \sum_{\substack{q \leqslant x \\ q \in P_\mu}} \frac{1}{q}.$$

Combining (12), (13) and (14), we complete the proof of the Lemma.

3. Proof of Theorem 1

Let B and $L_m(x, B)$ be as in the statement of Theorem 1. Then by (10) and the Lemma,

$$\begin{aligned}\sum_{\substack{n\leqslant x\\ d_m(n)\leqslant \exp L_m(x,B)}} d_m(n) &\leqslant \sum_{\substack{n\leqslant x\\ W_m(n)\leqslant L_m(x,B)}} d_m(n)\\ &\leqslant \sum_{n\leqslant x} d_m(n)\left\{\frac{W_m(n)-A(m)\log\log x}{B\sqrt{\log\log x}}\right\}^2\\ &\leqslant \frac{c_0}{B^2}\,x(\log x)^{m-1}\end{aligned}$$

for some constant c_0. This proves (2).

To prove (3), note only that

$$\begin{aligned}\sum_{\substack{n\leqslant x\\ d_m(n)\geqslant \exp L_m(x,B)}} 1 &\leqslant \exp(-L_m(x,B))\sum_{n\leqslant x} d_m(n)\\ &\ll x(\log x)^{m-1-A(m)}\exp(B\sqrt{\log\log x}),\end{aligned}$$

as required.

4. Remarks on the Proof of Theorem 2

To prove the second theorem we appeal to Theorems A and B of [5] (or Theorems 3.1.1 or 4.1.1 of [3]). These theorems require a long list of hypotheses (satisfied of course by our example above) and are cumbersome to state. We therefore content ourselves with a brief description of the basic elements of the theorems.

Suppose that $\sum_{n=1}^{\infty} b(n)\,n^{-s}$ is a Dirichlet series with $b(n)\geqslant 0$ that satisfies certain regularity assumptions including a functional equation with multiple gamme factors (in the sense of Chandrasekharan and Narasimhan [2]). Suppose further that there is a set S of positive integers of density zero which supports the average order of $b(n)$, i.e.,

$$\sum_{\substack{n\leqslant x\\ n\in S}} b(n)\sim \sum_{n\leqslant x} b(n) \qquad \text{as } x\to\infty.$$

Then Ω-estimates for the error terms in the Riesz sums $\sum_{n\leqslant x} b(n)(x-n)^{\rho}$ are determined in terms of the summatory function of the $b(n)$ (as in the classical methods of Hardy [6] and Szegö and Walfish [7], *et al.*) and also in terms the "thinness' of the set S. The thinner S is the better the estimate.

Furthermore, the $\pm$ character of the Ω-estimates is made precise, determined by relationships between the various parameters in the functional equation and ρ.

The introduction into these theorems of the dependence on the set S leads to the improvements in the results of Szegö and Walfisz [7] described in the remarks after Theorem 2 above. Our claim that Theorem 2 appears to be "best possible by present methods" is based on the observation that our set

$$S = \{n : d_m(n, K) \geqslant \exp L_m(x, B)\}$$

appears to be optimal. That is a thinner set would not provide enough support for the average order of $d_m(n, K)$.

5. Some Special Cases

One special case of interest is when $m = 1$ and $\rho = 0$. Put $E(x) = \sum_{n \leqslant x} a(n) - h\lambda x$, where h is the class number,

$$\lambda = 2^{r_1 + r_2} \pi^{r_2} R / (\omega \sqrt{|\Delta|}),$$

R is the regulator, ω is the number of roots of unity and Δ is the discriminant of K. (Here, $E(x) = \Delta_{1,0}(x)$.) Then, by Theorem 1, if $B = B(x)$ tends to ∞ with x,

$$\sum_{n \leqslant x}{}^{*} a(n) \sim h\lambda x,$$

and

$$\sum_{n \leqslant x}{}^{*} 1 \ll x (\log x)^{-A} \exp(B \sqrt{\log \log x})$$

where the $\sum_{n \leqslant x}^{*}$ is only over those n with $a(n) \geqslant (\log x)^{A} \exp(-B \sqrt{\log \log x})$. Theorem 2 implies that there exists a positive constant B such that

$$E(x) = \Omega^{**}\{(x \log x (\log \log x)^{A})^{(N-1)/(2N)} \exp(-B \sqrt{\log \log \log x})\}$$

where Ω^{**} is Ω_{+} if $N = 2$ or 3 and K is totally real, Ω_{-} if $N = 2$ and K is imaginary and $\Omega_{\pm}$ otherwise. This last result improves known Ω-estimates for $E(x)$ given by Szegö and Walfisz [7] by essentially a power of $\log \log x$.

In particular, if $K = \mathbb{Q}(\sqrt{-1})$, then $a(n) = r(n)/4$ where $r(n)$ is the number of representations of n as the sum of two squares. We then get the Ω-estimate in Gauss' circle problem

$$\sum_{n\leqslant x} r(n)-\pi x$$
$$=\Omega_{-}\{(x\log x)^{1/4}(\log\log x)^{(\log 2)/4}\exp(-B\sqrt{\log\log\log x})\}.$$

If $K=\mathbb{Q}$ and $m=2$ (so $A=0$), we have $d_m(n,K)=d(n)$, the divisor function of Dirichlet and

$$\sum_{n\leqslant x} d(n)-x(\log x+2\gamma-1)$$
$$=\Omega_{+}\{(x\log x)^{1/4}(\log\log x)^{(3+2\log 2)/4}\exp(-B\sqrt{\log\log\log x})\}.$$

These last two results were obtained by the author using other methods in [3, 4]. They improve Hardy's [6] classical results of 1916 by essentially a power of $\log\log x$.

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